A BASIS FOR A THEORY OF NONLOCAL ELASTIC FILTRATION USING THE EQUATIONS OF ELASTICITY

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The proposed [1-3] nonlocal formulation of the hypothesis that the ground pressure is constant in nonstationary pressure filtration in a deep elastic stratum is derived from the equilibrium equations for the stratum–roof system. The roof is considered to be a flat plate [4] and the floor of the stratum is assumed to be rigid. An equation is established for the scale of the region of influence on the stress and pressure distributions at a point.

1. In the theory of the nonlocal elastic filtration of a homogeneous fluid in a deep stratum [1-3], the linear equation for nonstationary filtration flow

$$(a_\rho + a)\frac{\partial p}{\partial t} - b\frac{\partial \sigma}{\partial t} = \frac{k_0}{m_0\mu_0}\nabla^2 p + \frac{G}{m_0\rho_0} \tag{1.1}$$

is complemented by the hypothesis (in integral form) that the ground pressure $\Gamma(x_1, x_2)$ is constant at each point of the stratum,

$$\sigma(x_1, x_2, t) + \int\limits_{-\infty}^{+\infty}\int \Phi\left(\frac{x_1-\xi_1}{d}, \frac{x_2-\xi_2}{d}\right) p(\xi_1, \xi_2, t)\, d\xi_1 d\xi_2 = \Gamma(x_1, x_2) \tag{1.2}$$

Here ρ, μ are the filtrating fluid density and viscosity; k, m are the stratum permeability and porosity; p is the interstitial pressure, σ is the effective stress in the stratum frame, and

$$\rho/\rho_0 = 1 + a_\rho(p - p_0), \qquad m/m_0 = 1 + a(p - p_0) - b(\sigma - \sigma_0)$$

where the subscript 0 indicates the unperturbed state; a_ρ, a, b, k_0 are constants; G is the source or sink intensity needed to imitate the operation of the drill hole; Φ is an influence function for which we can take the Gauss function

$$\Phi(x_1, x_2) = \frac{1}{\pi d^2}\exp\{-(x_1^2 + x_2^2)\}$$

If the scale d of the region of influence is much less than the typical dimension of the region in which p varies, Eq. (1.2) becomes the usual local condition that the ground pressure is constant [5]. But if d is relatively large, (1.2) reduces to the equation $d\sigma/dt = 0$, or $\sigma - \sigma_0 = 0$ [1, 2].

Equation (1.2) takes the following forms:

for the one-dimensional plane-parallel case

$$\sigma(x, t) + \int\limits_{-\infty}^{+\infty} \Phi\left(\frac{x-\xi}{d}\right) p(\xi, t)\, d\xi = \Gamma(x) \tag{1.3}$$

$$\Phi(x) = \frac{1}{d}G(x), \qquad G(x) = \frac{1}{\sqrt{\pi}}\exp(-x^2)$$

for the axisymmetric plane-radial case

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$$\sigma(r,t)+\int_0^\infty \Phi\left(\frac{r}{d},\frac{\rho}{d}\right)p(\rho,t)\,\rho d\rho=\Gamma(r) \tag{1.4}$$

$$\Phi(r,\rho)=\frac{1}{d^2}G(r,\rho),\qquad G(r,\rho)=2I_0(2r\rho)\exp\{-(r^2+\rho^2]\}$$

where I_0 is a Bessel function of zero order and imaginary argument.

2. Consider the one-dimensional plane-parallel case. The thickness of the rock lying on a porous stratum of thickness h will be simulated by an infinite elastic flat plate (Fig. 1) of rigidity D.

Ignoring the inertia term and letting w denote the displacement of the plate, we construct the bending equation for the plate

$$D\,d^4w\,/\,dx^4=\Gamma-\sigma-p \tag{2.1}$$

By Hooke's law

$$\sigma=Ew\,/\,h,\qquad w=\sigma h\,/\,E$$

where E is Young's modulus for the stratum frame.

We note that Eqs. (1.1) and (2.1), written in terms of w and the hydraulic head, were solved directly in [4] for the case of nonstationary flow to a gallery.

Because of analytic complexities the authors only considered the construction of an approximate equation for the size of the conical depression as a function of the time.

We transform (2.1) to an integral equation of the form (1.3). We put

$$\eta=\left(\frac{E}{hD}\right)^{1/4}x,\qquad u(\eta)=\frac{\partial\sigma}{\partial t},\qquad f(\eta)=-\frac{\partial p}{\partial t} \tag{2.2}$$

The Eq. (2.1) can be written as follows, after differentiating both sides with respect to t:

$$d^4u\,/\,d\eta^4+u=f \tag{2.3}$$

Using (2.3), we can express $u(\eta)$ in terms of $f(\eta)$. To do this we make a Fourier integral transformation [6], putting

$$U(\lambda)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}u(\eta)\,e^{i\lambda\eta}d\eta,\qquad F(\lambda)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}f(\eta)\,e^{i\lambda\eta}d\eta \tag{2.4}$$

Then from (2.3) we have

$$U(\lambda)=F(\lambda)\,/\,(\lambda^4+1) \tag{2.5}$$

We note that [7]

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}e^{-i\eta\lambda}\frac{d\lambda}{\lambda^4+1}=\sqrt{\frac{2}{\pi}}\int_0^\infty\frac{\cos(\eta\lambda)}{\lambda^4+1}d\lambda=\left(\frac{\pi}{2}\right)^{1/2}\exp\left(-\frac{|\eta|}{\sqrt{2}}\right)\sin\left(\frac{|\eta|}{\sqrt{2}}+\frac{\pi}{4}\right) \tag{2.6}$$

Applying the inverse Fourier transform and the convolution theorem [6] to (2.5), we have

$$u(\eta)=\frac{1}{2}\int_{-\infty}^{+\infty}\exp\left\{-\frac{|\eta-\xi|}{\sqrt{2}}\right\}\sin\left(\frac{|\eta-\xi|}{\sqrt{2}}+\frac{\pi}{4}\right)f(\xi)\,d\xi \tag{2.7}$$

Noting (2.2), after integration with respect to t, we obtain from (2.7):

$$\sigma(x,t)+\int_{-\infty}^{+\infty}\Phi_1\left(\frac{x-\xi}{\delta}\right)p(\xi,t)\,d\xi=\Gamma(x) \tag{2.8}$$

$$\Phi_1(x)=\frac{1}{\delta}G_1(x),\quad G_1(x)=\frac{1}{\sqrt{2}}e^{-|x|}\sin\left(|x|+\frac{\pi}{4}\right),\quad \delta=\left(\frac{4hD}{E}\right)^{1/4} \tag{2.9}$$

We note that $G_1(x)$ is normalized in $(-\infty,+\infty)$ and its numerical values almost coincide with those of $G(x)$, defined by (1.3).

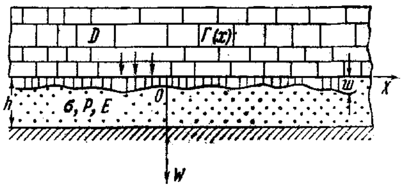

Fig. 1

It follows from (2.6) and (2.9) that

$$G_1(x) = \frac{\sqrt{2}}{\pi} \int_0^\infty \frac{\cos(\sqrt{2}\, x\lambda)}{\lambda^4 + 1} d\lambda \tag{2.10}$$

We approximate the function $(\lambda^4 + 1)^{-1}$ in (2.10) by the exponential function

$$(\lambda^4 + 1)^{-1} = \exp(-{}^1/_2 \nu^2 \lambda^2) \tag{2.11}$$

where the constant ν is defined below. Then, by (2.11), we obtain [7]

$$G_1(x) = \frac{1}{\sqrt{\pi}\,\nu} \exp \frac{-x^2}{\nu^2}$$

Thus, noting (2.9), we have

$$\Phi_1(x) = \frac{1}{\sqrt{\pi}\,(\nu\delta)} \exp \frac{-x^2}{\nu^2} \tag{2.12}$$

As a result, from (1.3) and (2.12), we have

$$\Phi\left(\frac{x}{d}\right) = \frac{1}{\sqrt{\pi}\, d} \exp \frac{-x^2}{d^2}, \qquad \Phi_1\left(\frac{x}{\delta}\right) = \frac{1}{\sqrt{\pi}\,(\nu\delta)} \exp \frac{-x^2}{(\nu\delta)^2} \tag{2.13}$$

It follows from (2.12) that the integral condition that the ground pressure is constant (1.13), proposed as a hypothesis, and Eq. (2.8), obtained from the equations of elasticity theory, coincide if we put

$$d = \nu\delta \tag{2.14}$$

We note that the choice of $\Gamma(x)$ on the right side of (2.8) is not significant for the solution of ordinary problems in filtration theory, since in the nonstationary case the important relation is between the time derivatives $\partial\sigma/\partial t$ and $\partial p/\partial t$ and not between the functions σ and p. Traditionally this function is interpreted as the ground pressure.

The constant ν in the approximation for $(\lambda^4 + 1)^{-1}$ can be found, for example, from the equation

$$\int_0^\infty \frac{d\lambda}{\lambda^4 + 1} = \int_0^\infty \exp \frac{-\nu^2\lambda^2}{2} d\lambda \tag{2.15}$$

Then ν and the unknown parameter d are given by

$$\nu = \frac{2}{\sqrt{\pi}}, \qquad d = 2\left(\frac{4hD}{\pi^2 E}\right)^{1/4} \tag{2.16}$$

3. Consider the axisymmetric plane-radial case. Let r denote the radial coordinate and let us retain the notation of Sec. 2.

In the axisymmetric case the bending equation for the plate has the form

$$D \frac{1}{r} \frac{\partial}{\partial r} \left\{ r \frac{\partial}{\partial r} \left[\frac{1}{r} \frac{d}{dr} \left(r \frac{\partial w}{\partial r} \right) \right] \right\} = \Gamma - \sigma - p \tag{3.1}$$
$$(\sigma = Ew/h, \qquad w = \sigma h/E)$$

As in the two-dimensional case, we transform (3.1) to an integral equation. We put

$$\eta \left(\frac{E}{hD}\right)^{1/4} r, \qquad u(\eta) = \frac{\partial \sigma}{\partial t}, \qquad f(\eta) = -\frac{\partial p}{\partial t} \tag{3.2}$$

After differentiating both sides with respect to t, we can write (3.1) as

$$\frac{1}{\eta} \frac{d}{d\eta} \left\{ \eta \frac{d}{d\eta} \left[\frac{1}{\eta} \frac{d}{d\eta} \left(\eta \frac{du}{d\eta} \right) \right] \right\} + u = f \tag{3.3}$$

We apply the Hankel integral transform [6] with kernel $J_0(\lambda, \eta)$, where J_0 is the Bessel function of zero order, to (3.3). Put

$$U(\lambda) = \int_0^\infty u(\eta) J_0(\lambda\eta)\, \eta\, d\eta, \qquad F(\lambda) = \int_0^\infty f(\eta) J_0(\lambda\eta)\, \eta\, d\eta \tag{3.4}$$

Multiplying both sides of (3.3) by $J_0(\lambda, \eta)\eta$ and integrating with respect to η from 0 to ∞, we obtain [7]

$$U(\lambda) = F(\lambda) \, / \, (\lambda^4 + 1) \tag{3.5}$$

Applying the inversion formula to (3.4) and noting (3.5), we have

$$u(\eta) = \int_0^\infty f(\xi)\, \xi \, d\xi \int_0^\infty \frac{J_0(\eta\lambda) J_0(\xi\lambda)}{\lambda^4 + 1} \, d\lambda \tag{3.6}$$

Noting (3.2), after integration with respect to t, we have, from (3.6):

$$\sigma(r, t) + \int_0^\infty \Phi_1\left(\frac{r}{\delta}, \frac{\rho}{\delta}\right) p(\rho, t)\, \rho \, d\rho = \Gamma(r) \tag{3.7}$$

$$\Phi_1(r, \rho) \frac{1}{\delta^2} G_1(r, \rho), \; G_1(r, \rho) = 2 \int_0^\infty \frac{J_0(\sqrt{2}\, r\lambda) J_0(\sqrt{2}\, \rho\lambda)}{\lambda^4 + 1} \, d\lambda \tag{3.8}$$

We note that $G_1(r, \rho)$ is normalized with respect to both arguments in $(0, \infty)$.

We approximate $(\lambda^4 + 1)^{-1}$ in (3.8) by the exponential function (2.11). Then from (3.8) we obtain [7]

$$G_1(r, \rho) = \frac{2}{\nu^2} I_0\left(\frac{2r\rho}{\nu^2}\right) \exp \frac{-(r^2 + \rho^2)}{\nu^2}$$

Thus, noting (3.8), we have

$$\Phi_1(r, \rho) = \frac{2}{(\delta\nu)^2} I_0\left(\frac{2r\rho}{\nu^2}\right) \exp \frac{-(r^2 + \rho^2)}{\nu^2} \tag{3.9}$$

As a result, from (1.4) and (3.9), we have

$$\Phi\left(\frac{r}{d}, \frac{\rho}{d}\right) = \frac{2}{d^2} I_0\left(\frac{2r\rho}{d^2}\right) \exp \frac{-(r^2 + \rho^2)}{d^2} \tag{3.10}$$
$$\Phi_1\left(\frac{r}{\delta}, \frac{\rho}{\delta}\right) = \frac{2}{(\nu\delta)^2} I_0\left(\frac{2r\rho}{(\nu\delta)^2}\right) \exp \frac{-(r^2 + \rho^2)}{(\nu\delta)^2}$$

From (3.10) we see that the hypothetical condition (1.4) and Eq. (3.7), derived here, coincide if we accept (2.14), and so we have, approximately, $\nu = 2/\sqrt{\pi}$. Then, by (3.8) and (3.11), the parameter d, as yet undetermined, is given by (2.16). Thus follows the important result that in both the plane-parallel and the plane-radial cases the parameter d has the same value under the nonlocal condition (1.2) that the ground pressure is constant.

We note that if the approximation constant ν is defined by the condition that the root mean square deviation of the function $(\lambda^4 + 1)^{-1}$ from $(-\nu^2\lambda^2/2)$ is a minimum, we have

$$V_{3/2}(\nu^2, 0) + \frac{1}{2\sqrt{\pi}\nu^3} = 0$$

where $V_{3/2}(x, y)$ is Lommel's function of two variables [7].

4. It follows from the above investigations that the integral condition (1.2), which states that the ground pressure is constant at each point of the stratum, previously introduced as a hypothesis using the Gauss function, agrees very well with Eqs. (2.8) and (3.7), which are obtained from the equilibrium equations for a porous stratum—thick rock system lying on a stratum.

The constant d in (1.2) is given, to a high degree of accuracy, by

$$d = 2\,(4hD \,/\, E)^{1/4}$$

In solving actual problems in the theory of nonlocal filtration, we can use the condition (2.8) or (3.7) that the ground pressure is constant, obtained from the equations of elasticity theory, the form of the kernel in the condition depending on whether the problem is plane-parallel or plane-radial. It is simpler to use (1.2), which contains the familiar Gauss function, and which states that the ground pressure is constant, as in [1.2], taking d to have the value given above.

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